CLASSIFICATION ACTIVITIES AND DEFINITION CONSTRUCTION AT THE ELEMENTARY LEVEL

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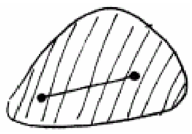
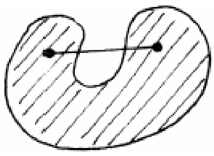
The French curricula strongly recommend activities involving tasks of comparison, reproduction, description, construction and representation in plane as well as three-dimensional geometry. There are now no specific guidelines concerning "classification's activities" regarding geometry. However, those are the very activities which lead pupils to explore a concept and then to identify mathematical properties useful for the characterization of objects of a given class. My epistemological aim is to propose a new point of view on the classification activities, that of the construction of definitions. The didactical implications of this perspective concern both the identification of classification processes through definition construction and the characterization of the guidance of such activities.

During an inaugural conference of "MATh.en.JEANS" (1992), the mathematician Berger talked about "convex things" as if they were human beings, in the following terms:



A "convex" is a person who is shaped in such a way that every time we take two points inside him any segment which joins them is inside.



You have here in front of you something which is clearly not convex. If you are keen on fractals, then forget it because "the convex" is definitely non-fractal. Convexity has a sort of security, control function: it guarantees that you have no hole, no hollow, and no warped line.

Let me focus on the above presentation of the concept of "convexity". It combines several features concerning the definition of "convex": an example and a counter-example are given in order to illustrate the mathematical definition, along with a morphological description of what is a convex figure. It is noteworthy that the "convex" concept can be grasped through four complementary and necessary ways: a definition couched in mathematical language, the illustration of the delimitation between convex figures and non-convex figures by an example and a counter-example (it will lead us to the etymological meaning of the word "definition", that is to say "delimitation"), a geometrical representation of the purpose and, in the end, a definition in common language. In order to achieve the full understanding of the current concept, we would still have to characterize a set of situations in which the concept of "convex" appears relevant and necessary (in Vergnaud's 1991 perspective of conceptual fields).

Through this example, I would like to underline the existing link between classifying process and defining process. To establish two classes amounts actually to delimitate a concept through what it is and what it is not. In this report, I shall consider a classification situation involving the difficult concept of “convexity” and analyze it through definitions construction. The results of an experiment, conducted at elementary school level with 10 year-old pupils, will be presented.

CLASSIFICATION AND DEFINITION CONSTRUCTION PROCESSES

How classification and definitions link up

There are two ways in which a definition can trap us. Firstly, we can delude ourselves into thinking that what can be easily expounded can be easily assimilated. Secondly, we can put too much trust in definitions because the latter are the result of a choice and thus show only one aspect of the concept. This is precisely what happens when a definition is presently axiomatically to a student.

Considering definitions as markers of the concept formation process gone through by the learner opens up a research avenue. In my introduction, I have underlined, the strong existing link between classifying and defining. I would also like to emphasize the importance of generalization and denomination processes in classifying. According to Hacking (1993), classifications and generalisations have to be linked. To use a name for one species amounts to producing generalisations and anticipations concerning the individual belonging to that species. Thus, using a common name to classify amounts to involve it in a projection process.

The prime importance of grasping characteristics of geometrical objects, during classification tasks, has been noticed by Freudenthal (1973) and Fletcher (1964). I shall take into account this view and propose a new reading of classifications tasks through definitions construction. Let me first recall the most common conceptions of the concept of definition in mathematics.

Commonly held views about mathematical definitions

Several researches have explored teachers’ conceptions relating to the concept of “definition” (Borasi 1992, Ouvrier-Buffet 2003a, Shir 2005). For instance, Zaslavsky and Shir underline that the features of a definition are commonly accepted as crucial:

> The imperative features relate to the following requirements: a mathematical definition must be *non-contradicting* (i.e., all conditions of a definition should co-exist), and *unambiguous* (i.e., its meaning should be uniquely interpreted). In addition, there are some features of a mathematical definition that are imperative only when applicable: A mathematical definition must be *invariant* under change of representation; and it should also be *hierarchical*, that is, it should be based on basic or previously defined concepts, in a *non-circular* manner (Zaslavsky & Shir, 2005, p.319)

More generally, I notice the following features: for teachers, at elementary and secondary levels,

- a definition should be minimal, non redundant (this is closely linked to a classical logical aspect: a mathematical theory has to be « well-formed » and a definition consists in a necessary and sufficient characterization of a concept). This is, in fact, a well-known conception: a definition should be useful and should have a specific place in proofs;
- to define is to give a name. Let me notice that the feature « denomination » is very present in teachers' discourses. However, to study definitions construction processes implies to reduce the place of the naming process, because the characterization of the concept itself comes first (I will come back on this fact below);
- a definition should state the existence (and also the essence) of a mathematical concept, in accordance to the Platonician view which maintains that a concept pre-exists to its definition;
- several linguistic features appear also, such as the following criteria: a definition should be precise, short, elegant, familiar and … universal;
- and, it is crystal clear that the way teachers spell out their exigencies about mathematical definitions is connected to teaching and learning: they actually underline that a definition should be based on anterior knowledge and should allow students to create their own mental image of a concept.

The question is now: how can we use these conceptions about the definition in order to design and manage classifications activities? Let me propose an exemplification.

A SITUATION ON "CONVEXITY"

The situation

The pupils (10 year old) have at their disposal physical objects, consisting in pieces of cardboard. It allows a manipulation of the objects. The geometrical figures are also given on a sheet (see figures below). The task is the following: "make two classes".

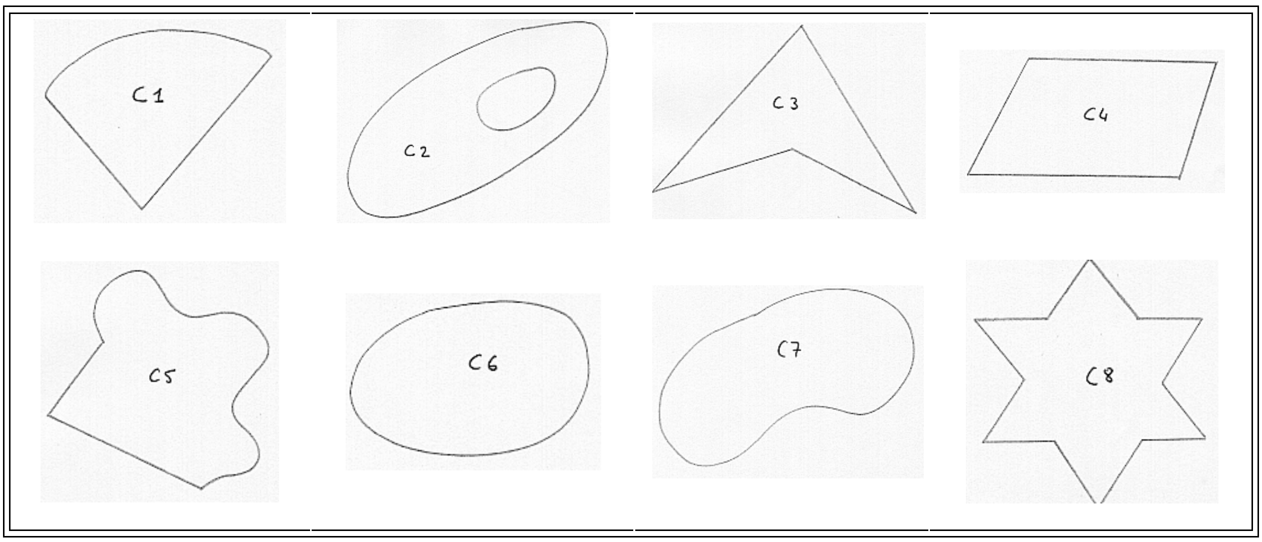


Figure 1: convex and non-convex figures, given to pupils.

The methodology

Five groups of 3-4 pupils took part into the activity. Concerning the progress of this activity, a MO (Manager Observer) has a specific place: his aim consists in orienting pupils' research to the construction of a definition of "convex", starting from classes produced by pupils. It implies that the MO has to use a particular command that is: the explicit demand of definition. In this perspective, he has to be particularly aware of the conceptions of definitions I have presented above.

A priori analysis

The concept at stake is "convex". According to Fletcher (1964), several definitions are conceivable, such as:

- Definition 1: a figure is convex if and even if, two points P and Q being given, all the points of the segment PQ belong to the figure.
- Definition 2: a figure is convex if and even if every straight line passing by any point included crosses the boundary in exactly two points.
- Definition 3: a figure is convex if and even if from each point of its boundary it passes at least one line of support.

Let me notice that a *dynamical* definition, similar to definition 3, can be stated, in common language: roaming the boundary of the figure, the whole figure is always at the "same side" (a direction for the roam being chosen). This kind of provisional definition should be evolved if logical and linguistic arguments are mobilized for instance (see Ouvrier-Buffet, 2003b).

- Definition 4: a figure is convex if and even if to each external point P to the figure corresponds one and only one point of the figure the nearest of P.

I have chosen the figures for the classification task according to the two following constraints. There is at most one figure with curve and non-curve lines in order to exclude a classification in accordance to "curve lines and non-curve lines" property from pupils' arguments. Quadrilaterals and other geometrical figures very institutionalized were outlawed in order to bypass pre-established classifications and definitions.

Steering a classification situation towards definition construction

In our perspective, it is necessary for the management to focus on the definition construction process, as a transversal competence and not as a final product.

An epistemological study of the concept of definition (Ouvrier-Buffet, 2003a&b, 2006) – a study which I can't report completely here – leads me to characterize some guiding styles acting on a definition construction process. Such a process is based on four poles, in relation with the kind of the considered situations. One of these poles concerns the construction of a theory (that does not concern us at the elementary level), another deals with heuristics and problem situations (that includes a specific work on examples and counter-examples), and the two other poles concern the logical as well as the linguistic aspects.

The MO can then manage pupils' progression, bearing in mind these several guidance elements. For instance, he can act taking into account that a definition is a specific statement: the MO may then formulate demands concerning logical and linguistic aspects of the current definition. The MO can also demand explicitly to pupils to generate examples and counter-examples. The latter give the opportunity to pupils to come back on the definition they are constructing.

It is worth stressing that the project on examples and counter-examples is not easy to implement at the elementary level. However, this heuristic approach is essential during a definition construction process. I underline thus the crucial dimension of working on examples and counter-examples in order to test a definition in particular, and in order to promote a scientific process in general.

Moreover, to take on board a relevant remark made by a pupil is a classical didactic guidance. Such a move assumes a major importance in the definition construction process: it sustains the devolution (in Brousseau's 1997 sense) of the definition construction process. We consider the devolution process to be active throughout the experiment thus avoiding the reduction of devolution to the terms of the problem itself and to the production of basic strategies (Brousseau, 1997 & Margolinas, 1993). If such a move is noteworthy, the one which consists in referring back the pupil to the prescribed task (writing a definition) is just as important.

PUPILS' STATEMENTS

Classes produced by pupils

The experiment described below was realized by the pupils only with cardboard figures. We can group theses several classifications into three categories: *morphological*, *mathematical* and *tiling*.

I mean by *morphological* every classifications involving physical descriptions of the manipulated forms. In every pupils' group, the two following classes appear:

- rounded / non-rounded: in one group, this classification leads pupils to construct orally the definition of a figure which is "more rounded than another one", mobilizing then considerations about the length of a curve and the area of a form;
- pointed / non-pointed.

I call *mathematical* the classifications mobilizing explicit anterior geometrical knowledge. The pupils explain four different *mathematical* classifications:

- figures having an axis of symmetry or not;
- polygon / non-polygon;
- figures having diagonals or not;
- figures having at most one angle and the others.

The category *tiling* corresponds to pupils' manipulations when they produced some kinds of "tangram puzzles". Pupils talked about figures which couple together or not.

They consider this classification as anecdotal and the vocabulary they use them laugh (the word “accoupler” has sexual connotations in French).

A group’s progress: the definitions produced

In this paragraph, I chose to focus on the way one group of pupils construct definitions. I shall underline in particular the pupils’ conceptions on the concept of definition and the guidance of the process.

This group has proposed successively three classifications:

- the figures having an axis of symmetry or not;
- the figures having at most one angle;
- and a classification very close to the concept of “convex” such as the following excerpt:

> When one connects the corners, the edges, it is interior or exterior.

This last classification leads pupils to elaborate two other classes, two figures being still unclassed (C2 – the holed piece – and C5 – the piece mixing straight curves and non-straight line). At this moment, pupils recall the instruction:

> It is not good because three columns are necessary and we have to make two classes.

The interventions of the MO felt into three distinct stages. Firstly, the MO recalls the instruction, that is to say recalls that we want to obtain two classes, then, pupils have to resolve the problem of C2 and C5. Secondly, the MO gives the name “convex”: this is connected to a philosophical view of definitions (i.e. to give a name before to characterize, in order to know what is about). Thirdly, the MO asks for a written definition of “convex”.

The pupils were quick to react, they looked up for the words in the dictionary which gave us a chance to point out that the way they relate to mathematical definitions is the same as the way they relate to lexical definitions, which does not apply to pupils in secondary schools. It becomes apparent then that the linguistic and logical levers can no longer be used. Moreover, pupils are content with one definition and the repeated questions of the MO are answered only because of the didactic contract. The MO can still use mathematical levers consisting in looking for characteristics of convexity: the explicit requests of examples and counter examples fall precisely within the latter category. The MO must be particularly alert to characteristic properties in terms of construction of definitions emerging in the pupils’ discourse.

Here are the successive definitions written by pupils: I am linking them to the MO’s interventions.

> Pupils’ definition 1: “convex: figure with points which connect on the inside”.

"point" is crossed and replaced by "angles" and then by "angles and round shapes".

> MO: what is the signification of “to connect a round shape”? Can you explain what a round shape is?

The MO asks then for an example and a counter-example of their first definition.

> Pupils' definition 2: "regular (or irregular) figures connecting together on the inside".

" irregular " is crossed.

The MO asks then another definition, excluding the idea of "lines on the inside". The pupils give a suitable reply:

> Pupils' answer: When one connects the points, it is interior. One cannot see how we can that in another way.

However, I notice that two other definitions could have emerged: definition 2 and definition 3.

Following on that demand, several definitions were written:

> Pupils' definition 3: "figure of whatever form, when we connect the two points, they are inside".

> Pupils' definition 4: "convex: when we draw diagonals, it stays inside the figure".

> Pupils' definition 5: "convex: when we link up a point with another, the straight line does not get out of the figure".

The MO then asks the pupils not to consider the segment but straight lines with the potentialities of definition 2 (presented in the a priori analysis) in mind.

The reader won't be surprised by the pupils' responses:

> But we are getting out of the theme! If we draw a straight line on all figures, they can all be convex!,

which, of course no longer complies with the prescribed task i.e. to set up two classes.

CONCLUDING REMARKS

There is a difference in nature between experiment about the definition constructions processes conducted at the primary level than and those conducted at the secondary level. Pupils at the elementary level have not yet a "culture" of mathematical definitions. This fact implies that the MO's freedom of action is somewhat limited. He cannot explore the whole range of guiding styles allowing a dialectic between definition construction and concept formation. However, the conceptual wealth being offered by defining situations built on classification tasks is promising. This paper illustrates the guidance possibilities of such situations, underlining in particular the guiding styles active in the defining process. Such experiments should be conducted again so as to fine-tune their impact on grasping of new concept at the elementary school.

References

Borasi, R. (1992) *Learning mathematics through inquiry*. Heinemann, New Hampshire.

Brousseau, G. (1997) *Theory of Didactical Situations in Mathematics*. Kluwer, Dordrecht.

Fletcher T.J. (1964) *Some lessons in Mathematics*. Cambridge University Press.

Freudenthal, H. (1973) *Mathematics as an educational task.* Reidel, Dordrecht.

Hacking, I. (1993) *Le plus pur nominalisme – l'énigme de Goodman*. (Trad. R.Pouivet). Editions de l'éclat.

Lakatos, I. (1961) *Essays in the logic of mathematical discovery*. Thesis, Cambridge (University Library).

Margolinas, C. (1993) *De l'importance du vrai et le faux dans la classe de mathématiques*. La Pensée Sauvage, Grenoble.

MATH.EN.JEANS (1992) Combien méchant peut être un convexe ? Ou quel est le convexe le moins rond ? *MATh.en.JEANS au Palais de la Découverte*, 167-172. Ed. MATh.en.JEANS, Paris.
(Web available : http://www.mjc-andre.org/pages/amej/edition/actes/actespdf/92167172.pdf)

Ouvrier-Buffet, C. (2003a) *Construction de définitions / construction de concept : vers une situation fondamentale pour la construction de définitions en mathématiques*. Leibniz, Grenoble. (Web available: http://tel.ccsd.cnrs.fr/)

Ouvrier-Buffet, C. (2003b) Can the Aristotelian and Lakatosian Conceptions of Definition Constitute a Tool for the Analysis of a Definition Construction Process? *Mediterranean Journal for Research in Mathematics Education, Vol2*, n°2, 19-36.

Ouvrier-Buffet, C. (to appear) Exploring Mathematical Definition Construction Processes. *Educational Studies in Mathematics*.

Vergnaud, G. (1991) La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques, n°10/2.3*, 133-169.

Zaslavsky, O. & Shir, K. (2005) Students' Conceptions of a Mathematical Definition. *Journal for Research in Mathematics Education. Vol36*, Issue 4, 317-346.