"RESEARCH" AS A WAY OF LEARNING NEW CONCEPTS: THE PARTICULAR CASE OF CONSTRUCTING DEFINITIONS

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We propose to consider new situations for the study of the formation of mathematical concepts. This paper is based upon mathematical research, and on the process of definition-construction in particular. We shall explain our choices and give an overall survey of the situations of definition-construction (called SDC) in which we have conducted experiments.

Foreword: the experiments described in this paper have been carried out with first year university students, but they are relevant to the high school students as well.

"DOING MATHEMATICS"

When the construction of a new concept is being discussed, professional mathematicians could point out the specificity of defining as a mathematical activity. However, although definition-construction has an obvious place in mathematical research, it seems that next to nothing has been written on the subject of "definition-construction" in the relevant literature, neither in pure mathematics, nor in mathematics education or in the different curricula. Indeed we could make the following remarks: firstly, Lakatos (1961) developed in parallel the formation of a concept and the construction of its definitions (using the method of proof and refutations). Secondly, the situations of classification (as proposed and analysed by Freudenthal (1973) as defining activities and by Mariotti & Fischbein (1997)) and redefining situations (as proposed by Borasi (1992), when redefining "circle" in Taxicab Geometry for instance) can be studied from the definition-construction point of view. But the classification and redefining tasks are only the tip of the iceberg consisting of "definition-construction situations". We will propose different types of situations of definition-construction (called SDC) from now on. Thirdly, we notice a specific point in French curricula, included in the expression "doing mathematics". Although there is no consensus about what "doing mathematics" means for mathematicians (it depends in fact on each individual), we can still underline the main features of our idea of "doing maths". Of course, it includes specific and transversal knowledge and skills[i], such as the following: proving, conjecturing, refuting, creating, modelling, extending but also transforming a questioning process, being capable of non-linear reasoning, having a scientific responsibility and above all defining. All these points may have a place in French curricula under the (unique) key word "scientific activity".

We assume that definition-construction situations represent an "opening" for the construction of concepts at school, which can be integrated in curricula. We have now to substantiate this fact, so, the aims of this paper are: to define SDC (Situations of Definition-Construction), to characterise mathematical objects propitious to SDC, to grasp "unanswerable" elements to bear out the potentiality of SDC for the formation of concepts.

We use the expression "definition-construction" naturally, but what is its exact mathematical meaning? What are the theoretical elements, required for the mathematical explanation and the design of such situations in the classroom?

CHARACTERISATION OF SDC & THEORETICAL BACKGROUND

At the beginning of our research, we started from a "simple" definition of SDC i.e. "situations whose resolution involved the construction of definitions". It means, in particular, that there is no predetermined solution to a certain type of problems, but an optimal resolution with definition-construction. The study on SDC seems to present a great complexity due both to the difficult concept of "definition" (from a mathematical point of view) and to the lack of SDC in curricula and textbook (from an educational point of view). We assume that a specific theoretical background is required for the study of such novel situations, in order to build and analyse both the definition-

construction and the concept-formation. It doesn't imply necessarily that theoretical tools, used by Freudenthal, Mariotti-Fischbein or Vinner for instance, should be dismissed, but the core of our research on SDC consists in proposing a theoretical background,

- which is appropriated to every type of SDC described below and to every field of mathematics (i.e. its use shouldn't be limited to classification or redefining tasks and to a certain mathematical field such as geometry),
- which can describe a defining process both from an mathematical and a didactical point of view,
- which can include students' pre-existing conceptions on definition (we assume that SDC should bring about an evolution of conceptions on the concept of definition).

Let us now specify key elements of characterisation of SDC and the theoretical tools we chose. An epistemological and mathematical study of the concept of "definition" leads us to a typology of SDC; we can identify three main types, which can be described as follows:

- CLASSIFICATION consists in delimiting what characterises a concept, starting from examples and counter-examples for instance.
- MATHEMATISATION/MODELLING: the characterisation of this type of SDC is initiated by its name. Let us propose an example: "define a mathematical object, which can represent the set of plants" (i.e. the elements of the whole vegetable kingdom). It could correspond to the mathematical concept of "tree".
- PROBLEM-SITUATION: this expression was used by Lakatos (1961) and was not exactly defined, but according to Lakatos, starting from a vague idea of a mathematical concept (such as Euler's formula for the study of polyhedra) can be enough for marking the beginning of a definitional procedure (Lakatos,1961, p.69). It comes close to our first natural definition.

The previous typology was established having regard to an identification of three emblematic conceptions: the Lakatosian, the Aristotelian and the Popperian conceptions. The choice of these conceptions was made in accordance with three main "levels" of the defining process i.e. dialectic between definitional procedure and concept-formation (Lakatos & Problem-Situation), language and logic (Aristotle & taxonomy tasks), and axiomatic (Popper & construction of new theories).

The characterisations of both typology and conceptions were written in order to plan, describe, grasp and explain the construction of definitions. That's why we used the cK¢ model[ii] (Balacheff): this model allows an identification of stages in the construction of definitions; it brings elements, which can help us grasp the formation of concepts, and it can include students' pre-existing conceptions on definition. We develop this theoretical background and the theorisation of the Lakatosian and Aristotelian conceptions in Ouvrier-Buffet (2002b&2004) (*).

Lakatos' work offers an interesting model of production of new knowledge, and his thesis discusses the complex dialectic between a definitional procedure (so definition-construction and concept-formation) and proof. In order to make a long story short, let us give a foretaste of this research. We remind here some key words of the promising Lakatosian view and their characterisation: *zero-definition*, *proof-generated definition*, *refutation*. We will present an overall characterisation of the Lakatosian conception with the cK¢ model. A conception is characterised, in the cK¢ model, by: a set of problems (here Problem-Situation), a set of *operators* and a set of *control-structures* (the last two undefined terms can be used in their common sense for a start), and a system of representation (in this case, the system of representation involved a mathematical research activity).
A *zero-definition* is a tentative definition emerging at the beginning of the investigation. It may evolve into a *proof-generated definition* or just disappear. It is brought about by proof and stands out as the most important notion in Lakatos' view: the product of *proof-generated definition* is directly linked to the type of SDC (i.e. Problem-Situation, according to Lakatos). Of course, the method of refutation is central to Lakatos' development. In particular, from the definition-construction viewpoint, several *operators* can be mobilized: they could consist in generating counter-examples, coming back to the generation of a proof, and, to a minor extent, making linguistic, logic and axiomatic demands: this last *operator* is specifically representative of both the Aristotelian and the Popperian conceptions. The Lakatosian set of *operators* contributes to the

evolution of a *zero-definition*. Hence, the notion of *zero-definition* is promising and useful for the characterisation of the start of a definitional procedure and its evolution. How does Lakatos *control* his definitional procedure? To answer this question presents a real complexity, in particular because both notions of *zero-definition* and *proof-generated definition* are defined by Lakatos each with regard to the other. Thus one *control-structure* of the definitional procedure mobilizes the proof. This aspect needs to be studied.

We complete this current characterisation of SDC by specificities of mathematical concepts involved in our experimented situations: we design three SDC with objects coming from discrete mathematics. Their main features are that these objects are accessible by their representations, they are not institutionalised[iii] (so the students have no a priori expectations when they work on these objects), and these objects can be worked on in different ways.

OUR SDC

We have experimented two main types of SDC. The first one was CLASSIFICATION: two situations were designed as described below. One of them involves the mathematical object "tree" (see Ouvrier-Buffet 2002a) and the other the object "discrete straight line". The main results concerning students' construction of definitions are mentioned below.
The second type of SDC was PROBLEM-SITUATION: the mathematical concepts at stake in such situations can be variable. A first experimentation was based on "displacements on a regular grid map" (see Ouvrier-Buffet 2002b for the description of this situation): it allows a natural problematisation of concepts of "generator" and "minimality" (it means a problematisation in Z of concepts generally taught in vector spaces). The second experimentation involves "discrete triangles": we propose to describe the main features and results of this experimentation compared with results from the other type of SDC CLASSIFICATION observed when watching the students.

SPECIFICATION OF STUDENTS' DEFINITION-CONSTRUCTION PROCEDURE

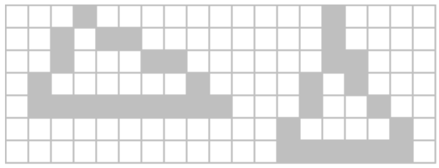
Classification with an explicit request of definition (tree and discrete straight line)

The statement of such a situation contains a question ("how could you define…?") and examples and counter-examples (identified as such or not) of a mathematical object.
The resolution by students of the two experimented SDC (type CLASSIFICATION) is made up of the following main elements: the major defining procedure used by students is based on the Aristotelian method of definition by *genus and differencia*. Moreover, students use given examples and counter-examples in order both to determine characteristics of "tree" or "discrete straight line" and to test their current definition. These comparison and test mechanisms seem natural because students' task consists in defining, delimiting a mathematical object starting from examples and counter-examples. Hence, a reflexivity on *zero-definitions* comes from refutations with counter-examples on one hand, and from linguistic and logical considerations on the other. We underline the fact that students are able to deal with different approaches to the mathematical object: and, they question the implication and the equivalences between the different points of view tackled.

Problem-situation (discrete triangle)

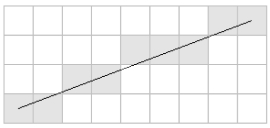
The SDC was proposed to two groups of students (first university year, scientific section) as follows: "one wants to colour squares on a regular grid map. Draw triangles (colouring squares). Explain your construction."
No explicit request of definition(s) was formulated. But such a situation carries the need of definition-construction (discrete triangles), with primitive elements (discrete straight line or segment). There are more ways than one to solve this problem-situation. It is possible to use different approaches to the mathematical discrete object and thus different ways to build it appear, e.g. starting from real straight line and one of its definitions or constructing such a discrete triangle starting from three given pixels (*).

Note that the perceptive aspect of discrete drawn objects is important and non-neutral in the students' debates we have observed. This picture shows how the perceptive round or hollow aspect of a triangle can play a part in a work on the definition and the construction of such discrete objects.



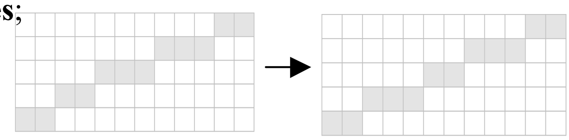
The problematic of construction is coupled with an axiomatic problematic i.e. to define discrete straight line requires questioning the existence (and uniqueness even) of the intersection of two discrete straight lines: this last point adds to the difficulty of the task a priori (*). We focus now on the geometrical object "discrete straight line".

Different *zero-definitions* of discrete straight line are conceivable: a first group of *zero-definitions* is connected to an approach of this discrete object by the real object. We have three *zero-definitions*:



Zdef1 : set of the pixels crossed by a real line;
Zdef2 : set of the pixels "**the nearest**" of a real line;
Zdef3 : set of the pixels "**inside**" a band.

Should you not draw another discrete straight line, when the above real straight line is given? This question underlines the potentialities for evolution of a *zero-definition* (among the above-mentioned). This mathematical analysis of potential evolution of conceivable *zero-definitions* is crucial when designing SDC. It determines and marks out ways for concept-formation.
Two other *zero-definitions* can be enounced, starting from a "regularity" view-point i.e. considering that the repartition of the pixels of a discrete straight line should be regular (a word to be defined! Again, this attests the potential evolution of the following *zero-definitions*):
Zdef4 : sequence of stages of pixels with specific **properties**;
Zdef5 : sequence of pixels stages with a uniform repartition, **non-improved** from the regularity viewpoint.



Remarks on observation of students' procedure (*)
Students' work on the construction of discrete triangles was very dependent on the perceptive aspect of discrete objects. Beyond this stage, different potential *zero-definitions* of "discrete straight line" were produced by students: they both use the "real straight line" and the "regularity" approaches. Students switch from one approach to another one. Eventually, only "regularity" problematic was retained, because students actually try to fit in within the discrete framework without any outside reference. That's very interesting when we consider the scientific approach. Nevertheless, students didn't work on equivalence between their potential *zero-definitions* and didn't identify the task as a "definitional task". Thus, all the *operators* in relation with linguistic and logical aspects were not mobilised and we notice that the evolution of *zero-definition* was blocked. In fact, students don't assume the responsibility of the construction of definition and they don't mention it. Nor do they engage in a test and a refutational process using counter-examples such as students working, or having worked, on classification tasks would do.

POTENTIALITIES OF SDC

The explicit request of definition in classification situations allows a reflexivity on the students' construction of definitions, in particular, the reflexivity on their current "definition" which causes this definition to evolve. SDC Classification constitute a good beginning for "definition-construction". In other respects, a Problem-Situation requires more time: it is a more difficult type of SDC and it seems necessary that a SDC type CLASSIFICATION precedes a PROBLEM-SITUATION in order to prepare students to a "first" contact with definition-construction search.
The main positive points of SDC are as follows:

- SDC are an opportunity of a work on scientific process (construction of definition and more: proof for instance). This scientific process is constituted by the students' exploration of different approaches, doubting, conjecturing, refuting (generating new counter-examples), testing etc.

- A work with SDC allows an enrichment (and generates modifications) of students' conception on the concept of definition in mathematics.
- SDC are a place for the formation of mathematical concepts: this fact is attested often by *zero-definitions* evolution, and not often by *proof-generated definition*.

Nevertheless, we can point to some limits to SDC. In particular, in the Problem-Situation, we have observed no evolution of potential *zero-definitions*, and the presence of definition-in-action were sufficient for a part of the resolution of the problem. We would like to point out that the explicit request for definition is pertinent in classification situations because it allows a reflexivity on definition-construction work through language and logic. But one of the main aims is for students to become "apprentice-mathematicians", to be able to mobilize scientific activity, so that they develop abilities and autonomy to construct definitions and mathematical concepts. Eventually, we have to suppress the explicit request for definition: hence, we have to design a set of SDC with a progression from the explicit request for definition in a classification situation to an open Problem-Situation without any indications about a "definitional task".

FURTHER OUTLOOK

At the moment, we have three main perspectives for further research. The first one consists in studying thoroughly proof-generated concept and *proof-generated definition* in order to build situation in which definition-construction is required (at the same time, such situations mobilize definition-construction and a reinvestment of produced definitions, in a dialectic interaction with a proof). The existence of this dialectical process is apparent in Lakatos' work but not described.
The second perspective concerns the extension of SDC to institutionalised concepts (and not only redefining situations, in a different context, with known concepts). The objects from the geometrical field seem to be worthwhile because they share common characteristics with discrete objects. We have some ideas to experiment (example: up to what extent can a convex (object) be "naughty"? Or how can we characterise the 'least round' convex shape?).
In order to integrate SDC in the curriculum as situations, which make a work on concept-formation possible, we have to precise how to conduct such situations. We don't specify in this paper the place and the role of the teacher, but we have elements for the characterisation of the authorized interventions of the teacher: neutrality and follow-up interventions (asking for a written definition, asking for new examples in order to focus students' research on a characterization, asking for counter-examples, and proposing counter-examples if (and only if) the situation is "blocked").

(*) ADDITIONAL INFORMATION IS AVAILABLE ON REQUEST

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[i] i.e. skills and knowledge which straddle mathematics, used in the whole variety of mathematical contexts.
[ii] "cK¢" are the initials of "**c**onception", "**k**nowledge" and "**¢**oncept".
[iii] An institutionalised concept is a "curriculum" concept i.e. a concept that has a place in the classic taught content.